Lecture 12

Monday, 3 October 2022 10:53 AM

Price of Anarchy in Network (Atomic) Congestion Games

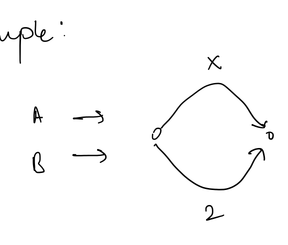
For a game Γ, $PoA(\Gamma) = \dfrac{\max\{\sum_i C_i(s) : s \text{ is an equilibrium}\}}{\min\{\sum_i C_i(s) : s \in \mathcal{S}\}}$ $= (OPT, s^*)$

(usu. for PNE, but could consider MNE, CCE, etc.)

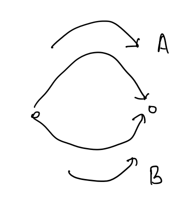
Given a class of games (e.g., NCGs)

$PoA = \max_{\text{instance } \Gamma} PoA(\Gamma)$

Example:

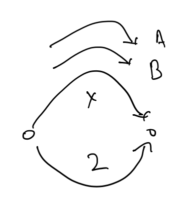


s^*, equilibrium



OPT = 3

s, equilibrium



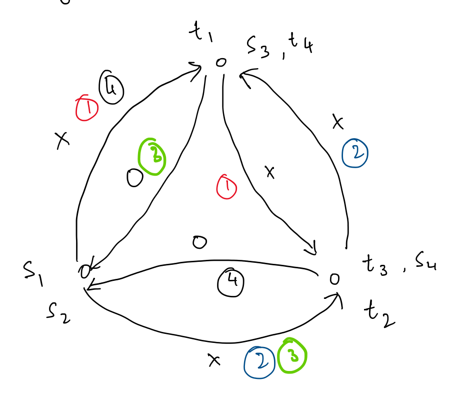
$\sum_i C_i(s) = 4$

hence $PoA(\Gamma) = 4/3$, $PoA \geq 4/3$.

Theorem: The PoA in NCGs w/ linear cost fns. is 5/2.

Lemma: $PoA \geq 5/2$

Proof (by example)



s^*: every player uses the single-edge s-t path,
OPT = 4 (this is also an equilibrium)

s (equilibrium): each player uses the 2-edge path

costs: ① 3, ② 3, ③ 2, ④ 2

total cost = 10

hence $PoA \geq \frac{10}{4} = \frac{5}{2}$

Lemma: $PoA \leq 5/2$

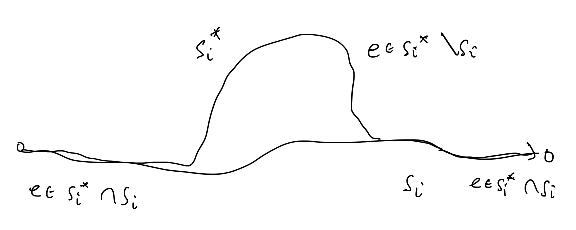
Proof: will use inequality: $x(y+1) \leq \frac{5}{3}x^2 + \frac{1}{3}y^2$

holds for $x, y \in \{0, 1, \dots\}$ (prove!)

doesn't hold for $x = \frac{1}{2}, y = 0$

Let s be an equilibrium, s^* be the min-cost strategy profile (i.e., $s^* \in \arg\min_{s \in \mathcal{S}} \sum_i C_i(s)$). Let $n_e(s), n_e(s^*)$ be the # of players using edge e in s, s^* respectively.

$$\forall i, \quad C_i(s) = C_i(s_i, s_{-i})$$
$$\leq C_i(s_i^*, s_{-i})$$
$$= \sum_{e \in s_i \cap s_i^*} c_e(n_e(s)) + \sum_{e \in s_i^* \setminus s_i} c_e(n_e(s)+1)$$



$$\leq \sum_{e \in s_i^*} c_e(n_e(s)+1)$$

Thus, $$\sum_i C_i(s) \leq \sum_i \sum_{e \in s_i^*} c_e(n_e(s)+1)$$
$$= \sum_e c_e(n_e(s)+1)\, n_e(s^*)$$
$$= \sum_e \left(a_e(n_e(s)+1) + b_e\right) n_e(s^*)$$
$$= \sum_e \left[a_e\, n_e(s^*)(n_e(s)+1) + b_e\, n_e(s^*)\right]$$
$$\leq \sum_e a_e\left(\frac{5}{3} n_e(s^*)^2 + \frac{1}{3} n_e(s)^2\right) + b_e\, n_e(s^*)$$
$$= \sum_e n_e(s^*)\left(\frac{5}{3} a_e\, n_e(s^*) + b_e\right) + \sum_e n_e(s)\left(\frac{1}{3} a_e\, n_e(s)\right)$$
$$\leq \frac{5}{3}\sum_e n_e(s^*)(a_e\, n_e(s^*) + b_e) + \frac{1}{3}\sum_e n_e(s)(a_e\, n_e(s) + b_e)$$

hence, $$\sum_i C_i(s) \leq \frac{5}{3}\sum_i C_i(s^*) + \frac{1}{3}\sum_i C_i(s)$$

$$\Rightarrow \sum_i C_i(s) \leq \frac{5}{2}\sum_i C_i(s^*)$$

$\Rightarrow \forall s, s^* \in \mathcal{S}$ s.t. s is an equilibrium,

$$\frac{\sum_i C_i(s)}{\sum_i C_i(s^*)} \leq \frac{5}{2}$$

Thus, for NCGs w/ linear cost fns., $PoA \leq 5/2$. ∎

(and hence, $PoA = 5/2$)

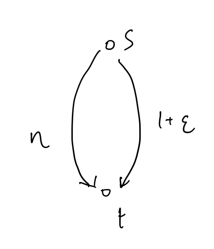
Global Connection Games

- directed graph $G = (V, E)$
- fixed $c_e \in \mathbb{R}_+$ on edges
- n players, each player has source s_i & destination t_i
- $\mathcal{S}_i = \{\text{set of } s_i - t_i \text{ paths}\}$
- $C_i(s) = \sum_{e \in s_i} c_e / n_e(s)$

i.e., cost of edge is divided equally among all players that use the edge.

Claim: $PoA \geq n$

Proof: n players



s^*: each player uses $1+\varepsilon$ edge, total cost $= 1+\varepsilon$

s: each player uses n edge, total cost $= n$

Claim: $PoA \leq n$

(prove yourself)

so instead, we study the PoS

Claim: GCG is a potential game w/

$\phi(s) = \sum_e c_e H(n_e(s))$ as the potential

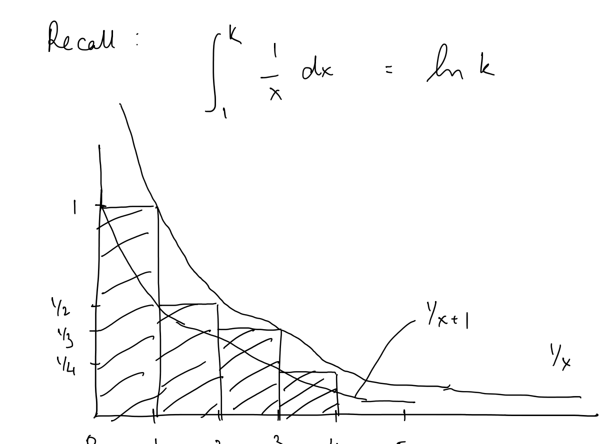
(where $H(k) = 1 + \frac{1}{2} + \dots + \frac{1}{k}$

and $H(0) = 0$.

$H(\cdot)$ is called the harmonic function)

Claim: $\ln(k+1) \leq H(k) \leq 1 + \ln k$

Proof: Recall: $\int_1^k \frac{1}{x}\, dx = \ln k$



$$\int_0^k \frac{1}{x+1}\, dx \leq H(k) \leq 1 + \int_1^k \frac{1}{x}\, dx$$

$$\ln(k+1) \leq H(k) \leq 1 + \ln k$$

Claim: In GCGs, $PoS \leq H(n)$

Proof: Let s^* be min-cost strategy. & s minimize $\phi(\cdot)$

Hence s is eq., $\phi(s) \leq \phi(s^*)$.

Further, $\forall s' \in \mathcal{S}$,

$$\phi(s') = \sum_{e: n_e(s') \geq 1} c_e H(n_e(s')) \geq \sum_{e: n_e(s') \geq 1} c_e = cost(s')$$

also, $$H(n)\, cost(s') = \sum_{e: n_e(s') \geq 1} c_e \cdot H(n) \geq \sum_{e: n_e(s') \geq 1} c_e \cdot H(n_e(s')) = \phi(s')$$

hence, $$cost(s^*) \geq \frac{\phi(s^*)}{H(n)} \geq \frac{\phi(s)}{H(n)} \geq \frac{cost(s)}{H(n)}$$

Hence, $PoS \leq H(n) \leq 1 + \ln n$ ∎

(note general form of argument for potential games:

if $\alpha\, cost(s) \leq \phi(s) \leq \beta\, cost(s)$,

then $PoS \leq \frac{\beta}{\alpha}$)

Claim: In GCGs, $PoS \geq H(n)$

Proof (by example):